Unified characterisations of resolution hardness measures

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Hardness measures for resolution

Historically first and best studied

- size of resolution proofs
- tree-like size of resolution proofs

Many ingenious techniques for size lower bounds

- feasible interpolation [Krajíček 97]
- size-width technique [Ben-Sasson & Wigderson 01]
- game-theoretic techniques [Pudlák & Impagliazzo 00, . . .]

Another central measure

- space of resolution [Esteban & Torán 99, . . .]
- lower bound method for space again via width [Atserias & Dalmau 08]

Why hardness measures?

Correspondence to SAT solvers

- size = running time
- space = memory consumption

What constitutes a good hardness measure?

- Which measure makes a formula hard/easy for a SAT solver?
- What is a good representation of boolean functions?
- How can this be best measured?

Hardness measures studied here for clause sets F

Size measures

- depth $\mathrm{dep}(F)$ of best resolution refutation of F
- hardness $\mathrm{hd}(F)$ (Horton-Strahler number)

Width measures

- (symmetric) width $\mathrm{wid}(F)$
- asymmetric width $\mathrm{awid}(F)$

Clause-space measures

- semantic space $\mathrm{css}(F)$
- resolution space $\mathrm{crs}(F)$
- tree-resolution space $\mathrm{cts}(F)$

Our objectives and contributions

Provide unified characterisations for hardness measures

- via Prover-Delayer games
- via partial assignments
- for arbitrary clause sets: unsatisfiable and satisfiable

This allows

- elegant proofs of basic relations between different hardness measures
- exact relations between the different measures
- generalised version of Atserias and Dalmau's result on the relation between resolution width and space

From unsatisfiable to satisfiable formulas

- Let h_0 be a measure for unsatisfiable clause sets, which does not increase by applying partial assignments.
- Extend h_0 to arbitrary clause sets F by

$$h(F) = \max\{\, h_0(F \upharpoonright_\alpha) \; : \alpha \text{ partial assignment, } F \upharpoonright_\alpha \text{ unsatisfiable} \,\}$$

Motivation

- understand performance of SAT solvers on satisfiable instances
- obtain 'good' SAT representations of boolean functions [Gwynne & Kullmann 13/14]
- 'good' = not too big and of good inference power
- all unsatisfiable instantiations should be easy for SAT solvers
- related notions in randomised context considered before [Achlioptas, Beame, Molloy 04] [Alekhnovich, Hirsch, Itsykson 05] [Ansótegui et al. 08]

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Size hardness measures: dep(F) and hd(F)

Depth

- dep(F) $=$ minimal height of a resolution tree for F

Hardness

- hd(F) $=$ height of the biggest full binary tree which can be embedded into each tree-like resolution refutation of F
- concept reinvented several times,
  e.g. as Horton-Strahler number of a tree

Basic relations

- $\mathrm{hd}(F) \leq \mathrm{dep}(F)$
- $2^{\mathrm{hd}(F)} \leq \text{tree-size}(F) \leq (\#\mathrm{var}(F) + 1)^{\mathrm{hd}(F)}$ [Kullmann 99] [Pudlák & Impagliazzo 00]

Width hardness measures: wid(F) and awid(F)

- width of a clause = # of its literals
- width of a proof = maximal width of its clauses

(Symmetric) width

- wid(F) = minimum width of a resolution refutation of F
- in each resolution step, both parents have width $\leq k$
- F needs to have width $\leq k$

Asymmetric width

- in each resolution step, one of the parents has width $\leq k$
- awid(F) = minimum k s.th. F has such a resolution refutation
- applies also to formulas with large width

Width vs. size

Short proofs are narrow

- seminal size-width technique

$$\mathrm{size}(F) = 2^{\Omega\left(\frac{(\mathrm{wid}(F) - \mathrm{initial\ width}(F))^2}{\#\mathrm{var}(F)}\right)}$$

[Ben-Sasson & Wigderson 01]

- generalises to asymmetric width

$$e^{\frac{\mathrm{awid}(F)^2}{8 \cdot \#\mathrm{var}(F)}} < \mathrm{size}(F) < 6 \cdot \#\mathrm{var}(F)^{\mathrm{awid}(F) + 2}$$

[Kullmann 04]

Game characterisations

Game-theoretic techniques for lower bounds

- classic Prover-Delayer game characterises hd(F) [Pudlák & Impagliazzo 00]
- asymmetric Prover-Delayer game characterises tree-size(F) [B., Galesi, Lauria 13]
- these games only work for unsatisfiable clause sets

Here

- a simplified Prover-Delayer game characterising hd(F) for arbitrary clause sets
- a game for asymmetric width awid(F)

Prover-Delayer game for hd(F)

- The two players play in turns. Delayer starts.
- Initially, the assignment θ is empty.
- A move of Delayer extends θ to $\theta' \supseteq \theta$.
- A move of Prover extends θ to $\theta' \supset \theta$ such that
  - θ' is a satisfying assignment for F, or
  - $\#\mathrm{var}(\theta') = \#\mathrm{var}(\theta) + 1$
- The game ends as soon as
  1. θ falsifies a clause in F, or
  2. θ satisfies F
- Delayer scores
  - as many points as variables have been assigned by Prover in case 1.
  - 0 points in case 2.

The characterisation

Theorem

There is a strategy of Delayer which can always achieve $\mathrm{hd}(F)$ *many points, while Prover can always avoid that Delayer gets more than* $\mathrm{hd}(F)$ *points.*

Sketch of proof

Strategy of Prover:

- If $F\restriction_{\theta}$ is satisfiable, then extend θ to a satisfying assignment.
- Otherwise choose x and $a \in \{0,1\}$ s.t. $\mathrm{hd}(F\restriction_{\theta\cup\{x=a\}})$ is minimal.

Strategy of Delayer:

- Initially choose θ such that $F\restriction_{\theta}$ is unsatisfiable and $\mathrm{hd}(F\restriction_{\theta})$ is maximal.
- For all other moves, if there are unassigned variables x and $a \in \{0,1\}$ with $\mathrm{hd}(F\restriction_{\theta\cup\{x=a\}}) \leq \mathrm{hd}(F\restriction_{\theta}) - 2$ extend θ by $x = 1 - a$. □

Extending the game to characterise asymmetric width

Key idea

- Prover can also forget some information.
- For simplicity, we only consider the unsatisfiable case.
- Can be extended to satisfiable clauses as in previous game.

The game

- The players play in turns. Delayer starts. θ is empty.
- Delayer extends θ to $\theta' \supseteq \theta$.
- Prover chooses some θ' compatible with θ such that $|\mathsf{var}(\theta') \setminus \mathsf{var}(\theta)| = 1$.
- The game ends as soon as θ falsifies a clause in F.
- Delayer scores the maximum of $\#\mathsf{var}(\theta')$ chosen by Prover.
- Prover must play in such a way that the game is finite.

Results

Theorem

- *There is a strategy of Delayer which guarantees* *at least* awid(F) *many points against every Prover.*
- *There is a strategy of Prover which guarantees* *at most* awid(F) *many points for every Delayer.*

Relation between the games

Consider the awid-game, when restricted in such a way that Prover must always choose some θ' with $\#\mathrm{var}(\theta') > \#\mathrm{var}(\theta)$. This game is precisely the hd-game.

Corollary

For all clause sets F *we have* $\mathrm{awid}(F) \leq \mathrm{hd}(F)$.

Characterisations by sets of partial assignments

Our starting point

Characterisation of width wid(F) by partial assignments
[Atserias & Dalmau 08]

We devise a hierarchy of conditions for

asymmetric width awid(F)	k-consistency
hardness hd(F)	weak k-consistency
depth dep(F)	bare k-consistency

Relation to games

- Sets of partial assignments give good Delayer strategies.
- Resolution proofs give good Prover strategies.

An example: asymmetric width

Definition

A set P of partial assignments for a clause set F is k-consistent if:

1. No $\varphi \in P$ falsifies F.
2. Let $\varphi \in P$ and x be a variable not assigned in φ.
   Then for all $\psi \subseteq \varphi$ with $\#\text{var}(\psi) < k$ and both $a \in \{0, 1\}$
   there is $\varphi' \in P$ with $\psi \cup \{x = a\} \subseteq \varphi'$.

Theorem

Let F be unsatisfiable. Then $\text{awid}(F) > k$ *if and only if there exists a k-consistent set of partial assignments for F.*

Space measures I

Semantic space

A semantic k-sequence for F is a sequence $F_1, \ldots, F_p$ such that:

1. $F_1 = \top$
2. for $i = 2, \ldots, p$, either $F_{i-1} \models F_i$ (inference), or there is $C \in F$ with $F_i = F_{i-1} \cup \{C\}$ (axiom download).
3. $\bot \in F_p$
4. $|F_i| \leq k$ for $i = 1, \ldots, p$

$\mathrm{css}(F) = \min\{\, k : F \text{ has a complete semantic } k\text{-sequence} \,\}$

Space measures II

Resolution space

A resolution k-sequence for F is a sequence $F_1, \ldots, F_p$ such that:

1. $F_1 = \top$
2. for $i = 2, \ldots, p$, either $F_i \setminus F_{i-1} = \{C\}$ where C is a resolvent of two clauses in F_i, or
   there is $C \in F$ with $F_i = F_{i-1} \cup \{C\}$ (axiom download).
3. $\bot \in F_p$
4. $|F_i| \leq k$ for $i = 1, \ldots, p$

$\mathrm{crs}(F) = \min\{\, k : F$ has a resolution k-sequence $\}$

Tree-resolution space

extra condition:

- If $\frac{C \quad D}{E}$ with $C, D \in F_{i-1}$ then $C, D \notin F_i$.

$\mathrm{cts}(F) = \min\{\, k : F$ has a tree k-sequence $\}$

Relations

Basic relations

For all clause sets F

- $\mathrm{css}(F) \leq \mathrm{crs}(F) \leq \mathrm{cts}(F)$ by definition
- $\mathrm{crs}(F) \leq 3\,\mathrm{css}(F) - 2$ similar to [Alekhnovich et al. 02]
- $\mathrm{cts}(F) = \mathrm{hd}(F) + 1$ [Kullmann 99]

Space and width

For an unsatisfiable CNF F of width r

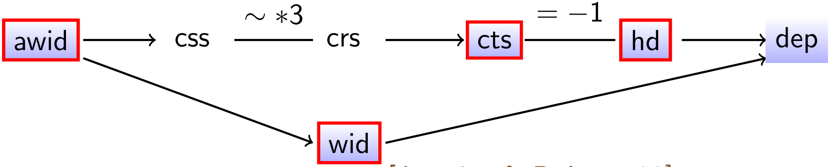
- $\mathrm{wid}(F) \leq \mathrm{crs}(F) + r - 1$ [Atserias & Dalmau 08]

A generalisation

For all clause sets F

- $\mathrm{awid}(F) \leq \mathrm{css}(F)$

Towards the full picture



[Atserias & Dalmau 08]

Characterisations

by Prover-Delayer games

by sets of partial assignments

Summary

Characterisations towards a unified framework for hardness measures

- via Prover-Delayer games
- sets of partial assignments
- for arbitrary clause sets: unsatisfiable and satisfiable

Main advantages

- elegant proofs of relations between hardness measures
- exact relations between the different measures

Open questions

Provide characterisations for

- semantic space
- resolution space

Exact relations

- Does $\mathrm{awid}(F) + 1 \leq \mathrm{css}(F)$ hold?
- Is $\mathrm{crs} = \mathrm{css}$?

Develop a general theory of hardness measures

- applicable to other proof systems than resolution