Solving Max SAT and #SAT on structured CNF formulas

Sigve Hortemo Sæther, Jan Arne Telle, Martin Vatshelle

University of Bergen

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Outline

- Equivalence of assignments

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- CNF formulas with few equivalence classes

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- Algorithm for MAX SAT

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- Polynomial time solvable cases

Equivalence of assignments to CNF formulas

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- The number of equivalence classes is called the **ps-value**.

- We need two standard definitions before we give a sufficient condition for low ps-value.

Incidence graph

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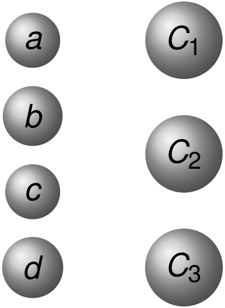

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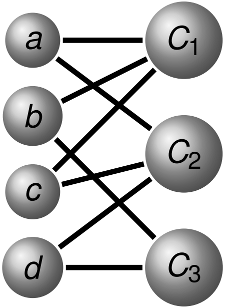

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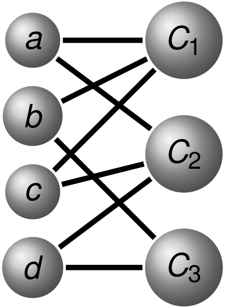
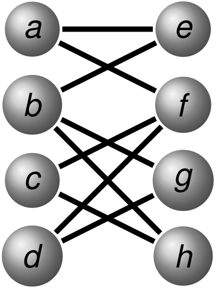


- For a CNF formula F we denote the incedence graph by $I(F)$.

Induced matching

Given a bipartite graph M is an induced matching if:



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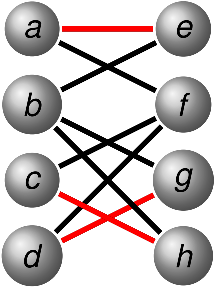

1. *M* is a matching.

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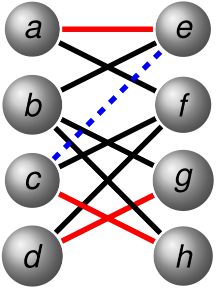

1. M is a matching.
2. No other edge is adjacent to 2 edges in M.

CNF formulas with low ps-value

Lemma

Let F be a CNF formula and k be the maximum size of an induced matching in $I(F)$.
The ps-value of F is at most $|\texttt{cla}(F)|^k + 1$

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Sketch of proof

1. *Let U be the set of unsatisfied clauses, and L the variables appearing in some clause of U, then there is a unique assignment for L.*
2. *There exist $U' \subseteq U$ of size at most k such that L is uniquely defined by U'.*
3. *There is at most $|\texttt{cla}(F)|^k + 1$ choices for U'*

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DP algorithm strategy

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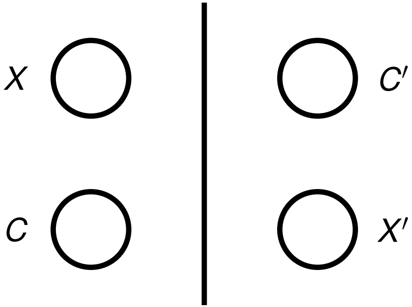
Note: the linear ordering can be replaced by a tree-like decomposition.

Equivalence over a cut

Let F be a CNF formula, $X \subseteq var(F)$ and $C \subseteq cla(F)$ defines a cut.



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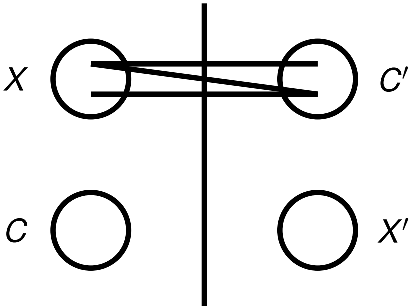

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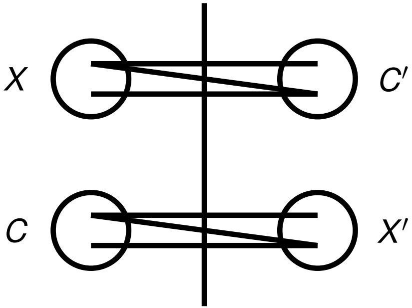

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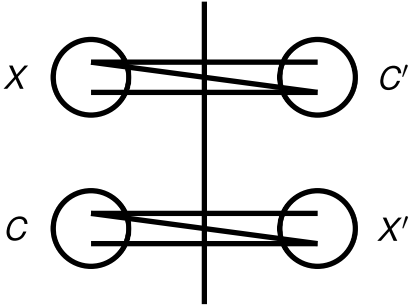

We define **ps-value of a cut** as the ps-value of the formula where each variable is removed from all clauses on the same side as the variable.

PS-width

1. The **ps-width** of a decomposition (ordering) is the max ps-value over all cuts defined by the decomposition.
2. The **ps-width** of a CNF formula is the min ps-width over all decompositions.

Runningtime for MAX SAT

Theorem

Given a formula F over n variables and m clauses and of total size s, and a decomposition of F of $\mathtt{ps}$-width k, we solve #SAT, and weighted MAXSAT in time $\mathcal{O}(k^3 s(m+n))$.

If the decomposition is a linear order the runningtime can be improved by a factor k.

Formulas of linear PS-width

We say a formula has an interval order if:

- Each variable and clause can be assigned an interval of the real line.
- Such that a variable x is in a clause c if and only if the interval of x intersects the interval of c.

Incidence graphs of such formulas are called interval bigraphs.

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Many other classes of bipartite intersection graphs correspond to formulas which have polynomial ps-width.

Relation to graph decomposition

- A tree-decomposition of $I(F)$ of tree-width $O(log(n))$ can be turned into a decomposition of polynomial ps-width.
- A clique-decomposition of $I(F)$ of constant clique-width can be turned into a decomposition of polynomial ps-width.

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- A clique-decomposition of $I(F)$ of constant clique-width can be turned into a decomposition of polynomial ps-width.
- An active community study a wide range of width parameters such as:
  tree-width, branch-width, rank-width, boolean-width, clique-width, cut-width, MM-width, MIM-width ...
  Bounding any of these parameters would prove polynomial ps-width.

Future research

1. Can we approximate ps-width?
2. Can we recognize graphs of MIM-width 1?
3. Does real world SAT instances have low ps-width?

THANK YOU