Proof Complexity and the Kneser-Lovász Theorem

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Outline

Kneser's conjecture and proof(s)

Lower bounds on proof complexity

Cases with efficient proofs: $k = 2$ and $k = 3$

Conclusions and Future Work

Kneser's Conjecture

- Stated in 1955 (Martin Kneser, Jaresbericht DMV)
- Let $n \geq 2k - 1 \geq 1$. Let $c : \binom{n}{k} \to [n - 2k + 1]$. Then there exist two disjoint sets A and B with $c(A) = c(B)$.
- $k = 1$ PHP!
- $k = 2, 3$ combinatorial proofs (Stahl, Garey & Johnson)
- $k \geq 4$: only proved in 1977 (Lovász) using algebraic topology.
- Combinatorial proofs known (Matousek, Ziegler). "hide" Alg. Topology
- No "purely combinatorial" proof known

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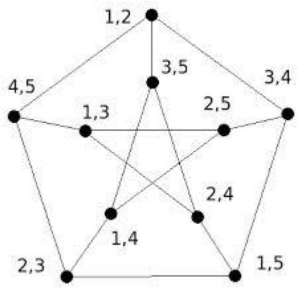
Kneser's Conjecture (cont.)

- the chromatic number of a certain graph $Kn_{n,k}$ (at least) $n - 2k + 2$. (exact value)
- Vertices: $\binom{n}{k}$. Edges: disjoint sets.
- E.g. $k = 2$, $n = 5$: Petersen's graph has chromatic number (at least) three.



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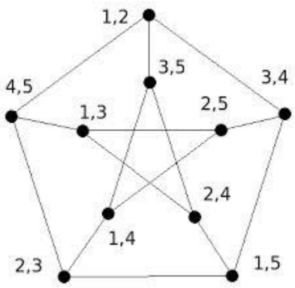

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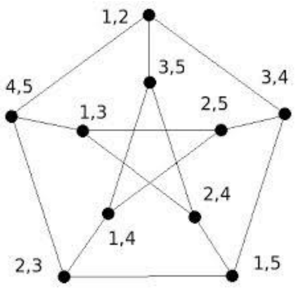
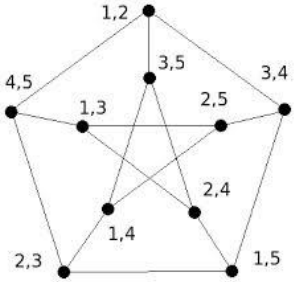


Kneser's conjecture: stronger form - Schrijver's Theorem

- inner cycle in Petersen's graph already chromatic number three.
- $A \in \binom{n}{k}$ stable if it doesn't contain consecutive elements i, $i+1$ (including $n, 1$).
- Schrijver's Theorem: Kneser's conjecture holds when restricted to stable sets only.



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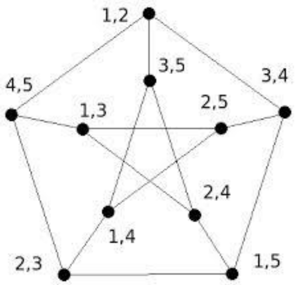

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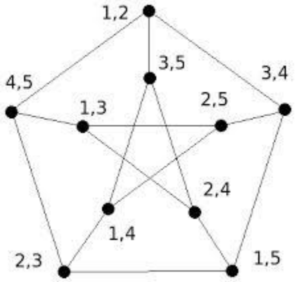

$Kneser_{k,n}$ as a Propositional Formula

- naïve encoding $X_{A,k} = TRUE$ iff A colored with color k.
- $X_{A,1} \vee X_{A,2} \vee \ldots \vee X_{A,n-2k+1}$ "every set is colored with (at least) one color"
- $\overline{X_{A,j}} \vee \overline{X_{B,j}}$ $(A \cap B = \emptyset)$ "no two disjoint sets are colored with the same color"
- Fixed k: $Kneser_{k,n}$ has poly-size (in n).
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- Proof systems
  - Frege - propositional sequent calculus,
  - Extended Frege - renaming allowed.
- Frege proofs:
  - PHP poly-size Extended Frege proofs (induction); poly-size Frege proofs via a counting argument (Buss).
  - ... but does $Kneser_{k,n}$ have such proofs?
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  Provide hard examples?
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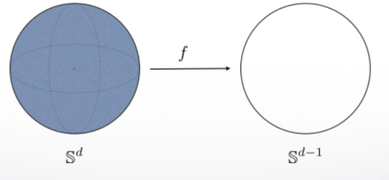
- Borsuk-Ulam: cannot map continuously and antipodally *n*-dim. sphere into a sphere of lower dimension.



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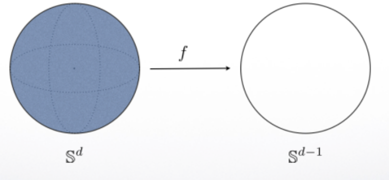

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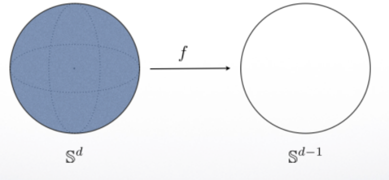

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Algebraic Topology: Trials and Tribulations (cont.)

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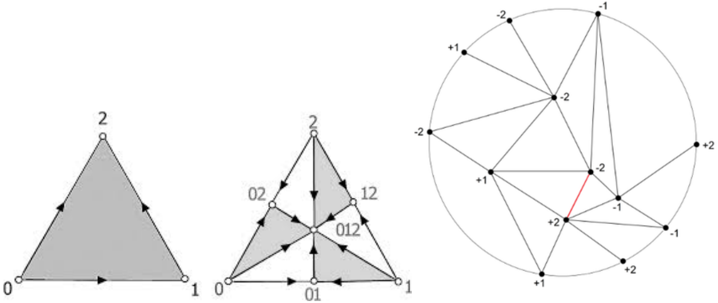

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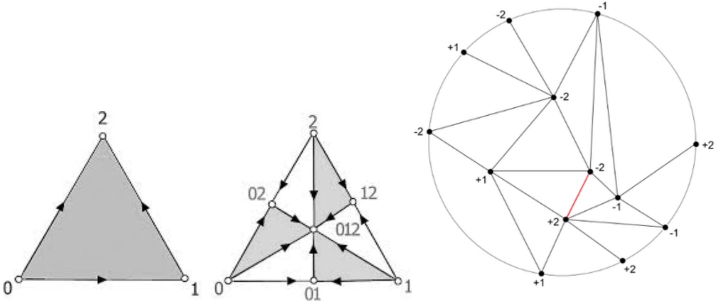

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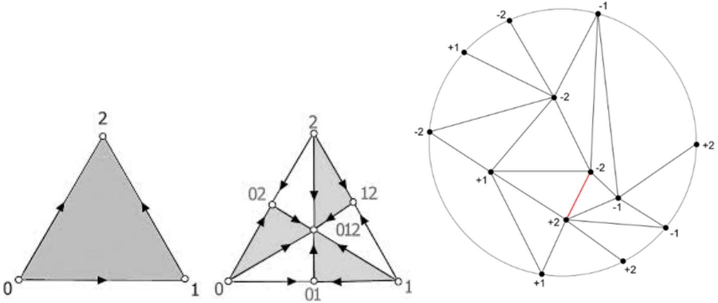

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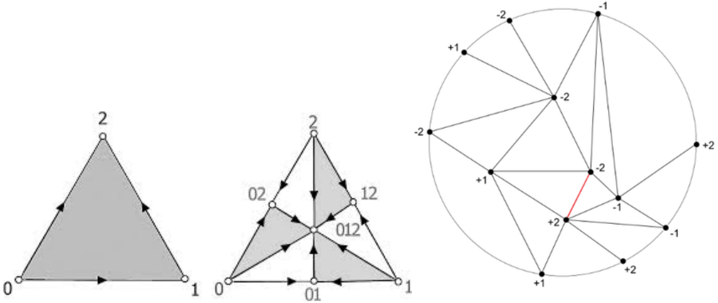

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Results in this paper

- $Kneser_{k+1,n}$ reduces to (is a special case of) $Kneser_{k,n-2}$.
- Thus all known lower bounds that hold for PHP (resolution, bd. Frege) hold for any $Kneser_k$.
- Cases with combinatorial proofs:
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Kneser's conjecture and proof(s)

Lower bounds on proof complexity

Cases with efficient proofs: $k = 2$ and $k = 3$

Conclusions and Future Work

Reduction from $Kneser_{k+1,n}$ to $Kneser_{k,n-2}$

- There exists a variable substitution $\Phi_k : Var(Kneser_{k+1,n}) \to Var(Kneser_{k,n-2})$ s.t. $\Phi_k(Kneser_{k+1,n})$ consists precisely of the clauses of $Kneser_{k,n-2}$ (perhaps repeated and in a different order)
- Let $A \in \binom{n}{k+1}$. Define $\Phi_k(X_{A,i})$ by:
  - **Case 1:** $A_{\leq k} \subseteq [n-2]$: $\Phi_k(X_{A,i}) = Y_{A_{\leq k},i}$
  - **Case 2:** $A_{\leq k} \not\subseteq [n-2]$: $(n-1, n \in A)$
    Let $A = P \cup \{n-1, n\}$, $|P| = k-1$. Let $\lambda = max\{j : j \leq n-2, j \notin P\}$. Define $\Phi_k(X_{A,i}) = Y_{P \cup \{\lambda\},i}$
- Clause $X_{A,1} \vee X_{A,2} \vee \ldots \vee X_{A,n-2k+1}$ maps to $Y_{B,1} \vee Y_{B,2} \vee \ldots \vee Y_{B,n-2k+1}$, $B = A$ (Case 1).
- Clauses $\overline{X_{A,i}} \vee \overline{X_{B,i}}$ $(A \cap B = \emptyset)$ map to $\overline{Y_{C,i}} \vee \overline{Y_{D,i}}$
- Case 2 cannot happen for both A and B. By case analysis $C \cap D = \emptyset$.
- Schrijver: similar, but check that mapping preserves stability.

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- Clause $X_{A,1} \vee X_{A,2} \vee \ldots \vee X_{A,n-2k+1}$ maps to $Y_{B,1} \vee Y_{B,2} \vee \ldots \vee Y_{B,n-2k+1}$, $B = A$ (Case 1).
- Clauses $\overline{X_{A,i}} \vee \overline{X_{B,i}}$ $(A \cap B = \emptyset)$ map to $\overline{Y_{C,i}} \vee \overline{Y_{D,i}}$
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- Schrijver: similar, but check that mapping preserves stability.

Reduction from $Kneser_{k+1,n}$ to $Kneser_{k,n-2}$

- There exists a variable substitution $\Phi_k : Var(Kneser_{k+1,n}) \to Var(Kneser_{k,n-2})$ s.t. $\Phi_k(Kneser_{k+1,n})$ consists precisely of the clauses of $Kneser_{k,n-2}$ (perhaps repeated and in a different order)
- Let $A \in \binom{n}{k+1}$. Define $\Phi_k(X_{A,i})$ by:
  - **Case 1:** $A_{\leq k} \subseteq [n-2]$**:** $\Phi_k(X_{A,i}) = Y_{A_{\leq k},i}$
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$k = 2$ (semantic) proof

- Basic result: Given any $(n-3)$-coloring c of $\binom{n}{2}$ and color $1 \leq l \leq n-3$, at least one of the following alternatives is true:
  1. there exist two disjoint sets $D, E \in c^{-1}(l)$.
  2. $|c^{-1}(l)| \leq 3$.
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- Now we count:
  - $p_r = |\{1 \leq \lambda \leq r : |c^{-1}(\lambda)| \geq 4 \text{ and } \bigcap_{A \in c^{-1}(\lambda)} A \neq \emptyset\}|$,
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  - Sequences M_r, N_r are monotonically increasing.
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- Buss counting formulas: binary encodings of the number of variables set to TRUE.
- can be simulated by Frege proofs.
- Examples:
  - For $n \geq 5$ and $1 \leq l \leq n-3$,

$$\bigvee_{\substack{D,E\in\binom{n}{2}\\ D\cap E=\emptyset}} (X_{D,l}\wedge X_{E,l}) \vee [Count((X_{S,l})_{S\in\binom{n}{2}}) \leq 3] \vee \bigvee_{i\in[n]} \Big(\bigwedge_{i\notin S} \overline{X_{S,l}}\Big).$$

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$k = 3$, polynomial EF

- For any $1 \leq \lambda \leq n - 5$ at least one of the following is true:
  1. $c^{-1}(\lambda)$ contains two disjoint sets
  2. $|c^{-1}(\lambda)| \leq 3n - 8$, or
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- Try counting:
  1. Count p_r, number of sets $c^{-1}(l)$, $1 \leq l \leq t$ such that $|c^{-1}(\lambda)| \geq 3n - 7$ (implicitly $\cap_{A \in c^{-1}(l)} A \neq \emptyset$).
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Kneser's conjecture and proof(s)

Lower bounds on proof complexity

Cases with efficient proofs: $k = 2$ and $k = 3$

Conclusions and Future Work

From Kneser-like results to hard SAT instances ?

- $2^{\Omega(n)}$ resolution complexity. Are they hard in practice ?
- $Kneser_{4,9}$ has $\binom{9}{4} \cdot (9 - 2 \cdot 4 + 1) = 252$ variables
- Want: small formulas.
- Schrijver ? Dolnikov's Theorem ?
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- Kneser, stable Kneser graphs: symmetries well understood.
  But: reason for unsatisfiability is more global.

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Further work

- Other proof systems: e.g. cutting planes (k=2), polynomial calculus, etc.
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Thank you! Questions ?